Basic linear algebra recap. Convergence rates.

Seminar

Optimization for ML. Faculty of Computer Science. HSE University

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- $\langle A, B\rangle = \text{tr}(A^T B)$

Convergence rate

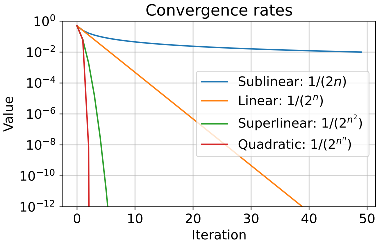

Figure 1: Illustration of different convergence rates

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$$r_k \leq Cq^k, \quad 0 < q < 1, C > 0$$

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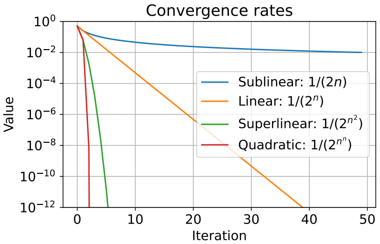

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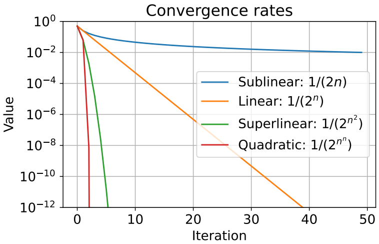

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- Any convergent sequence, that is slower (faster) than any linearly convergent sequence has sublinear (superlinear) convergence
- The infimum of all $0 \leq q < 1$ such that $r_k \leq Cq^k$ is called the **constant of linear convergence**, and q^k is called the **rate of convergence**.

Root test

Let $\{r_k\}_{k=m}^{\infty}$ be a sequence of non-negative numbers, converging to zero, and let

$$q = \lim_{k \to \infty} \sup_k \; r_k^{1/k}$$

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- The case $q > 1$ is impossible.

Ratio test

Let $\{r_k\}_{k=m}^{\infty}$ be a sequence of strictly positive numbers converging to zero. Let

$$q = \lim_{k \to \infty} \frac{r_{k+1}}{r_k}$$

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- If q does not exist, but $q = \lim\limits_{k \to \infty} \sup_k \dfrac{r_{k+1}}{r_k} < 1$, then $\{r_k\}_{k=m}^{\infty}$ has linear convergence with a constant not exceeding q.

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- In particular, if $q = 0$, then $\{r_k\}_{k=m}^\infty$ has superlinear convergence.
- If q does not exist, but $q = \lim_{k\to\infty} \sup_k \frac{r_{k+1}}{r_k} < 1$, then $\{r_k\}_{k=m}^\infty$ has linear convergence with a constant not exceeding q.
- If $\lim_{k\to\infty} \inf_k \frac{r_{k+1}}{r_k} = 1$, then $\{r_k\}_{k=m}^\infty$ has sublinear convergence.
- The case $\lim_{k\to\infty} \inf_k \frac{r_{k+1}}{r_k} > 1$ is impossible.
- In all other cases (i.e., when $\lim_{k\to\infty} \inf_k \frac{r_{k+1}}{r_k} < 1 \leq \lim_{k\to\infty} \sup_k \frac{r_{k+1}}{r_k}$) we cannot claim anything concrete about the convergence rate $\{r_k\}_{k=m}^\infty$.

Problem 1. Stupid important idea on matrix computations.

Suppose, you have the following expression

$$b = A_1 A_2 A_3 x,$$

where the $A_1, A_2, A_3 \in \mathbb{R}^{3 \times 3}$ - random square dense matrices and $x \in \mathbb{R}^n$ - vector. You need to compute b.

Which one way is the best to do it?

1. $A_1 A_2 A_3 x$ (from left to right)

Check the simple code snippet after all.

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3. It does not matter
4. The results of the first two options will not be the same.

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Problem 2. Connection between Frobenius norm and singular values.

Let $A \in \mathbb{R}^{m \times n}$, and let $q := \min\{m, n\}$. Show that

$$\|A\|_F^2 = \sum_{i=1}^{q} \sigma_i^2(A),$$

where $\sigma_1(A) \geq \ldots \geq \sigma_q(A) \geq 0$ are the singular values of matrix A. Hint: use the connection between Frobenius norm and scalar product and SVD.

Problem 3. Know your inner product.

Simplify the following expression:

$$\sum_{i=1}^{n} \langle S^{-1} a_i, a_i \rangle,$$

where $S = \sum_{i=1}^{n} a_i a_i^T, a_i \in \mathbb{R}^n, \det(S) \neq 0$

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- $r_k = \frac{1}{k^{10}}$
- $r_k = 0.707^k$
- $r_k = 0.707^{2^k}$

Problem 5. One test is simpler, than another

Determine the convergence or divergence of the following sequence:

$$r_k = \frac{1}{k^k}$$

Problem 6. Super but not quadratic.

Show, that the following sequence does not have a quadratic convergence.

$$r_k = \frac{1}{3^{k^2}}$$

LoRA: Low-Rank Adaptation of Large Language Models (arXiv:2106.09685)

Since current LLMs are too big to fit into the memory of the average user, we need to use some tricks to make them smaller. One of the most popular tricks is LoRA (Low-Rank Adaptation of Large Language Models).

Suppose we have matrix $W \in \mathbb{R}^{d \times k}$ and we want to perform the following update:

$$W = W_0 + \Delta W.$$

The main idea of LoRA is to decompose the update ΔW into two low-rank matrices:

$$W = W_0 + \Delta W = W_0 + BA, \quad B \in \mathbb{R}^{d \times r}, A \in \mathbb{R}^{r \times k},$$
$$rank(A) = rank(B) = r \ll \min\{d, k\}.$$

Check the notebook for the example implementation of LoRA.

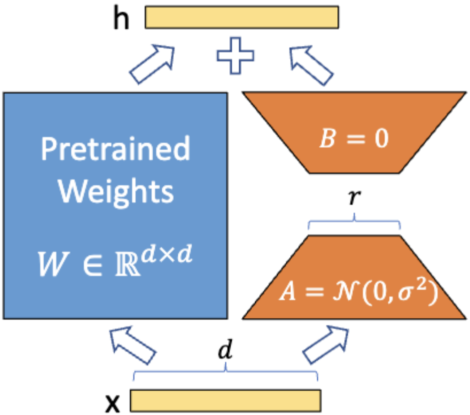

Figure 2: Illustration of LoRA