Gradient Descent. Convergence for quadratics; smooth convex case; PL case

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Gradient Descent

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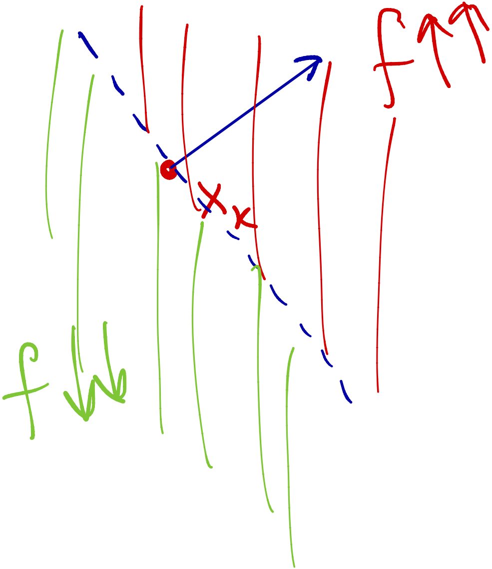

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$$|\langle f'(x), h \rangle| \leq \|f'(x)\|_2 \|h\|_2$$
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The result of this method is

$$x_{k+1} = x_k - \alpha f'(x_k)$$

Gradient flow ODE

Let's consider the following ODE, which is referred to as the Gradient Flow equation.

$$
\frac{dx}{dt} = -f'(x(t)) \qquad \text{(GF)}
$$

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$$
\frac{x_{k+1} - x_k}{\underbrace{t_{k+1} - t_k}_{t}}
$$

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and discretize it on a uniform grid with α step:

$$\boxed{\frac{x_{k+1} - x_k}{\alpha} = -f'(x_k),}$$

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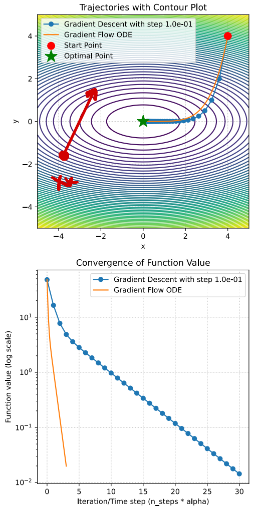
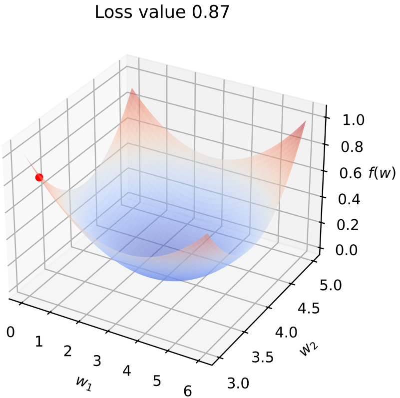
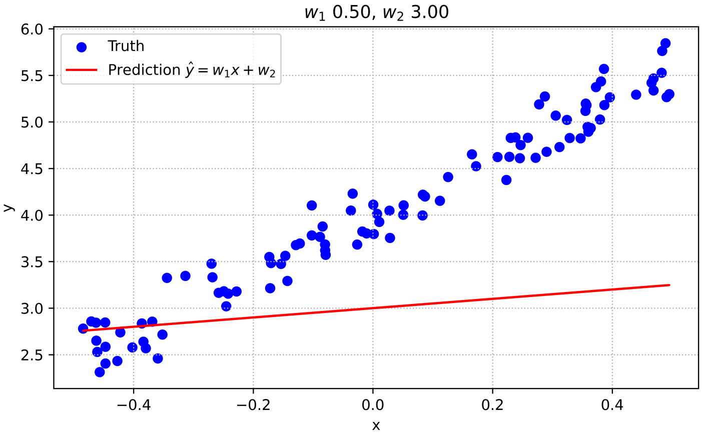


Figure 1: Gradient flow trajectory

Convergence of Gradient Descent algorithm

Heavily depends on the choice of the learning rate α:





Exact line search aka steepest descent

Метод наискорейшего спуска

$$\alpha_k = \text{arg}\min_{\alpha \in \mathbb{R}^+} f(x_{k+1}) = \text{arg}\min_{\alpha \in \mathbb{R}^+} f(x_k - \alpha \nabla f(x_k))$$

More theoretical than practical approach. It also allows you to analyze the convergence, but often exact line search can be difficult if the function calculation takes too long or costs a lot. An interesting theoretical property of this method is that each following iteration is orthogonal to the previous one:

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$$x_{k+1}(\alpha) = x_k - \alpha \nabla f(x_k)$$

Optimality conditions:

$$f(x_{k+1}(\alpha))$$

$$\frac{\partial f}{\partial \alpha} = \frac{\partial f}{\partial x_{k+1}}^T \frac{\partial x_{k+1}}{\partial \alpha} = 0$$

$$\nabla f(x_{k+1})^T \quad -\nabla f(x_k)$$

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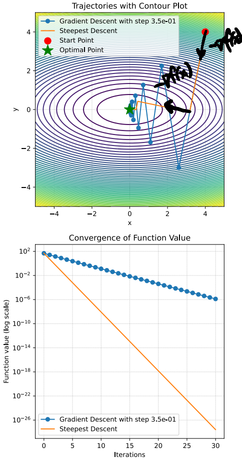

Figure 2: Steepest Descent

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Strongly convex quadratics

Coordinate shift

Consider the following quadratic optimization problem:

$$
\min_{x \in \mathbb{R}^d} f(x) = \min_{x \in \mathbb{R}^d} \frac{1}{2} x^\top A x - b^\top x + c, \text{ where } A \in \mathbb{S}^d_{++}.
$$

$O(n^2)$

$Ax^* = b$

$x^* = A^{-1} b$

$O(n^3)$

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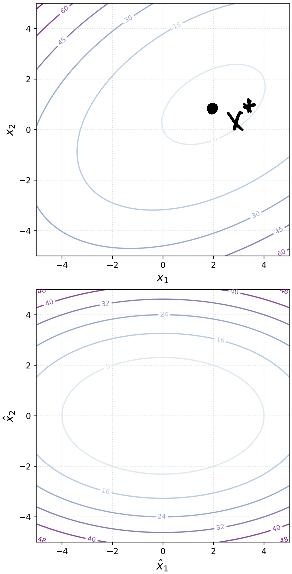

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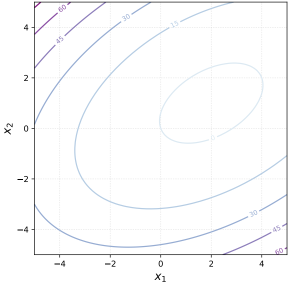
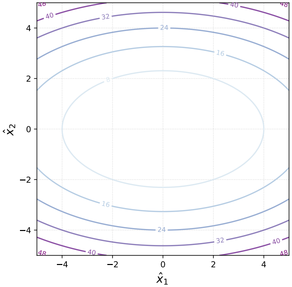
$$\boxed{A = Q \Lambda Q^T}$$





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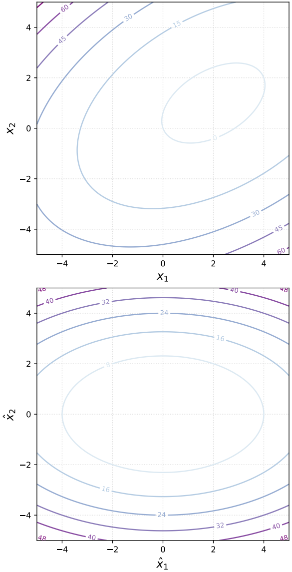
поворот

сдвиг



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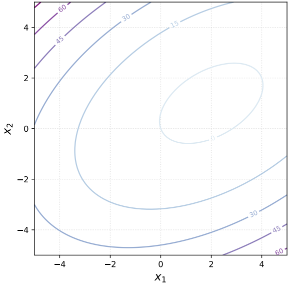

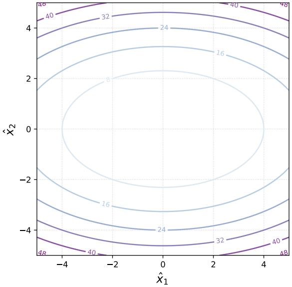

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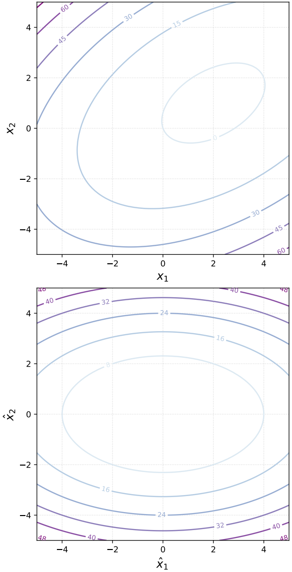
$$= \frac{1}{2}\hat{x}^T Q^T A Q \hat{x} + (x^*)^T A Q \hat{x} + \frac{1}{2}(x^*)^T A (x^*)^T - b^T Q \hat{x} - b^T x^*$$



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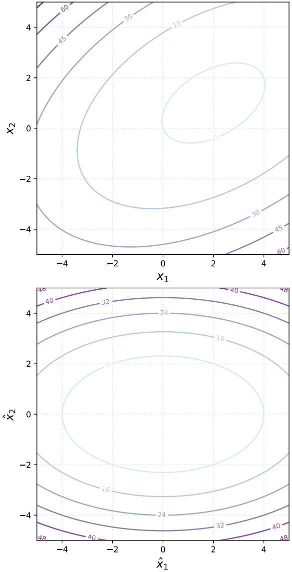
$$\begin{aligned} f(\hat{x}) &= \frac{1}{2}(Q\hat{x} + x^*)^\top A (Q\hat{x} + x^*) - b^\top (Q\hat{x} + x^*) \\ &= \frac{1}{2}\hat{x}^T Q^T A Q \hat{x} + (x^*)^T A Q \hat{x} + \frac{1}{2}(x^*)^T A (x^*)^T - b^T Q \hat{x} - b^T x^* \\ &= \frac{1}{2}\hat{x}^T \Lambda \hat{x} \quad + \text{const} \end{aligned}$$



Convergence analysis

просто вместо $\tilde{x}$ пишу x

Now we can work with the function $f(x) = \frac{1}{2}x^T\Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

$$x^{k+1} = x^k - \alpha^k \nabla f(x^k) = x^k - \alpha^k \cdot \Lambda x^k = \left(I - \alpha^k \Lambda\right) x^k$$

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x^{k+1}_{(i)} = (1 - \alpha^k \lambda_{(i)}) x^k_{(i)} \quad \text{For } i\text{-th coordinate}
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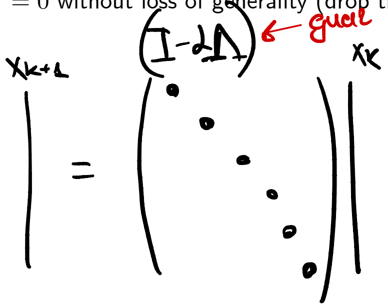

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$$\boxed{x^{k+1}_{(i)} = (1 - \alpha^k \lambda_{(i)})^k x^0_{(i)}}$$

если $\alpha^k = \alpha = \text{const}$

сходится $\quad |1 - \alpha^k \lambda_i| < 1$

необходимое условие сх-ти

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Let's use constant stepsize $\alpha^k = \alpha$. Convergence condition:

$$
\rho(\alpha) = \max_i |1 - \alpha\lambda_{(i)}| < 1
$$

← спектральный радиус матрицы $I - \alpha\Lambda$

Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

$|1 - \alpha\mu| < 1$ $\quad$ $|1 - \alpha L| < 1$

$-1 < 1 - \alpha\mu < 1$ $\quad$ $-1 < 1 - \alpha L < 1$

$0 < 2 - \alpha\mu < 2$ ✓ $\quad$ $0 < 2 - \alpha L < 2$ ✓

$2 - \alpha\mu > 0$ $\quad$ $2 - \alpha L > 0$

$\alpha < \frac{2}{\mu}$ $\quad$ $\alpha < \frac{2}{L}$

$\mu = 1$ $\quad$ $L = 10$

$\alpha < \frac{2}{1}$ $\quad$ $\alpha < \frac{2}{10}$

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$$
\begin{aligned}
|1 - \alpha\mu| &< 1 \\
-1 < 1 - \alpha\mu &< 1
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$$
\begin{aligned}
&|1 - \alpha\mu| < 1 \\
&-1 < 1 - \alpha\mu < 1 \\
&\alpha < \frac{2}{\mu} \qquad \alpha\mu > 0
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Remember, that $\lambda_{\text{min}} = \mu > 0, \lambda_{\text{max}} = L \geq \mu$.

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\begin{aligned}
|1 - \alpha\mu| &< 1 \\
-1 < 1 - \alpha\mu &< 1 \\
\alpha < \frac{2}{\mu} \qquad & \alpha\mu > 0
\end{aligned}
$$

$$
|1 - \alpha L| < 1
$$

Convergence analysis

Now we can work with the function $f(x) = \frac{1}{2}x^T\Lambda x$ with $x^* = 0$ without loss of generality (drop the hat from the $\hat{x}$)

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необх. усл. сх-ти

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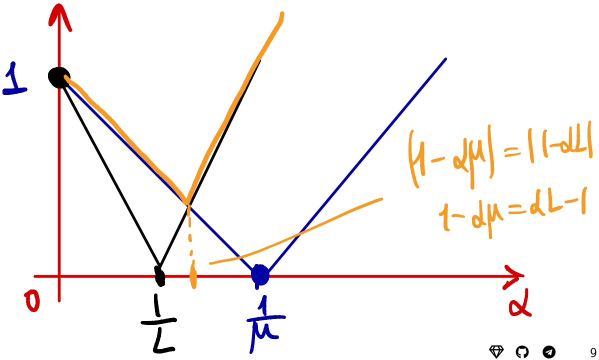

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\boxed{\alpha^* = \frac{2}{\mu + L}}
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↑ скорость сходимости

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\end{aligned}
$$

$$
x^{k+1} = \left(\frac{L - \mu}{L + \mu} \right)^k x^0 \quad f(x^{k+1}) = \left(\frac{L - \mu}{L + \mu} \right)^{2k} f(x^0)
$$

$$
x^{k+1} = \left(\frac{L/\mu - 1}{L/\mu + 1} \right)^k x_0 = \left(\frac{\varkappa - 1}{\varkappa + 1} \right)^k x_0
$$

Convergence analysis

So, we have a linear convergence in the domain with rate $\frac{\kappa-1}{\kappa+1} = 1 - \frac{2}{\kappa+1}$, where $\kappa = \frac{L}{\mu}$ is sometimes called condition number of the quadratic problem.

κ	ρ	Iterations to decrease domain gap 10 times	Iterations to decrease function gap 10 times
1.1	0.05	1	1
2	0.33	3	2
5	0.67	6	3
10	0.82	12	6
50	0.96	58	29
100	0.98	116	58
500	0.996	576	288
1000	0.998	1152	576

Polyak-Lojasiewicz smooth case

Polyak-Lojasiewicz condition. Linear convergence of gradient descent without convexity

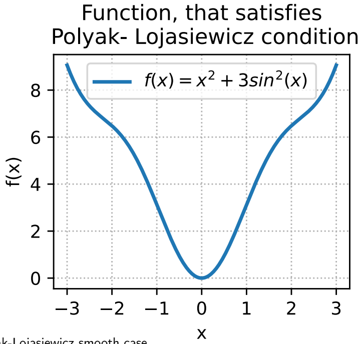
PL inequality holds if the following condition is satisfied for some $\mu > 0$,

$$\boxed{\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*)} \quad \forall x$$

It is interesting, that the Gradient Descent algorithm might converge linearly even without convexity.

The following functions satisfy the PL condition but are not convex. Link to the code

$$f(x) = x^2 + 3\sin^2(x)$$



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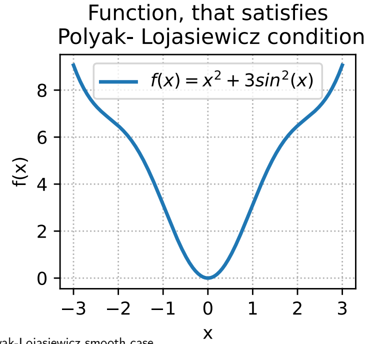
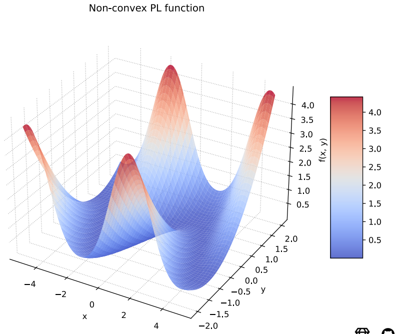
$$f(x,y) = \frac{(y - \sin x)^2}{2}$$





Convergence analysis

L - гладкость =

= липшицевость градиента

$\|\nabla f(x) - \nabla f(y)\|_2 \leq L\|x-y\|_2$ $\quad \|\nabla^2 f(x)\|_2 \leq L$

Theorem

Consider the Problem

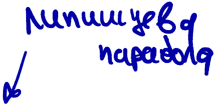
$$f(x) \to \min_{x \in \mathbb{R}^d}$$

and assume that f is μ-Polyak-Lojasiewicz and L-smooth, for some $L \geq \mu > 0$.
Consider $(x^k)_{k \in \mathbb{N}}$ a sequence generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying $0 < \alpha \leq \frac{1}{L}$. Then:

линейная сх-ть

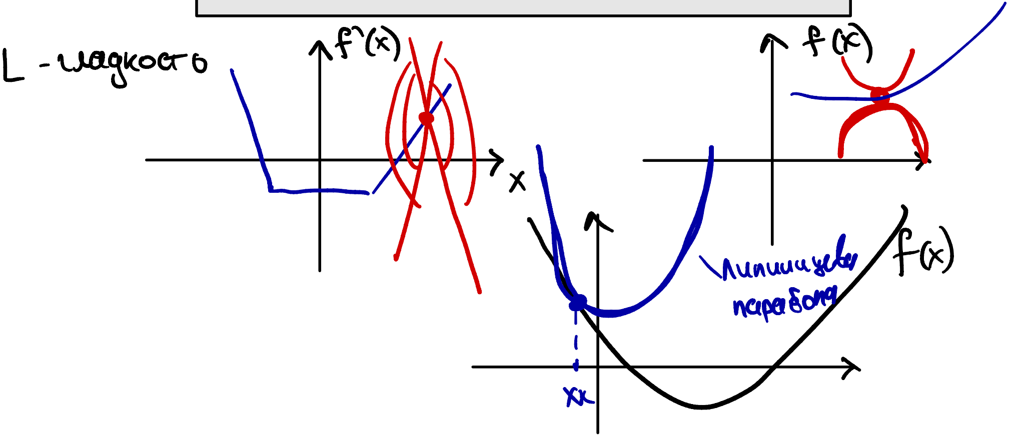
$$f(x^k) - f^* \leq (1 - \alpha\mu)^k (f(x^0) - f^*).$$

Convergence analysis



We can use L-smoothness, together with the update rule of the algorithm, to write

$$f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2$$



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&= f(x^k) - \alpha\|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2
\end{aligned}
$$

$$x^{k+1} = x^k - \alpha \nabla f(x_k)$$

$$x^{k+1} - x^k = -\alpha \nabla f(x_k)$$

Convergence analysis

по условию теоремы

$\alpha \leq \frac{1}{L}$

$\alpha L \leq 1$

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f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\
&= f(x^k) - \alpha\|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2 \\
&= f(x^k) - \frac{\alpha}{2}\left(2 - L\alpha\right)\|\nabla f(x^k)\|^2
\end{aligned}
$$

Convergence analysis

We can use L-smoothness, together with the update rule of the algorithm, to write

$$
\begin{aligned}
f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\
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&\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2,
\end{aligned}
$$

$-(2-\alpha L) \leq -1$

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&\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2,
\end{aligned}
$$

градиентный спуск -
- монотонный

Convergence analysis

$\alpha L \leq 1$

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$$
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&\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2,
\end{aligned}
$$

PL:

$$-\|\nabla f(x_k)\|_2^2 \leq -2\mu (f(x_k) - f^*)$$

where in the last inequality we used our hypothesis on the stepsize that $\alpha L \leq 1$.

$0 < 1 - \alpha\mu < 1$

$$f(x^{k+1}) \leq f(x_k) - \frac{2\alpha\mu}{2}(f(x_k) - f^*)$$

$$f(x^{k+1}) - f^* \leq f(x_k) - f^* - \alpha\mu(f(x_k) - f^*)$$

$$f(x^{k+1}) - f^* \leq (1 - \alpha\mu)(f(x_k) - f^*) = (1-\alpha\mu)^k (f(x) - f^*)$$

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&\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2,
\end{aligned}
$$

where in the last inequality we used our hypothesis on the stepsize that $\alpha L \leq 1$.

We can now use the Polyak-Lojasiewicz property to write:

$$
f(x^{k+1}) \leq f(x^k) - \alpha\mu(f(x^k) - f^*).
$$

The conclusion follows after subtracting f^* on both sides of this inequality and using recursion.

Any μ-strongly convex differentiable function is a PL-function

Theorem

If a function $f(x)$ is differentiable and μ-strongly convex, then it is a PL function.

Proof

By first order strong convexity criterion:

$$
f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2} \|y - x\|_2^2
$$

Putting $y = x^*$:

$$
f(x^*) \geq f(x) + \nabla f(x)^T (x^* - x) + \frac{\mu}{2} \|x^* - x\|_2^2
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&= \left(\nabla f(x)^T - \frac{\mu}{2}(x^* - x)\right)^T (x - x^*) = \\
&= \frac{1}{2}\left(\frac{2}{\sqrt{\mu}}\nabla f(x)^T - \sqrt{\mu}(x^* - x)\right)^T \sqrt{\mu}(x - x^*) =
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Let $a = \frac{1}{\sqrt{\mu}}\nabla f(x)$ and
$b = \sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}}\nabla f(x)$

By first order strong convexity criterion:

$$f(y) \geq f(x) + \nabla f(x)^T(y - x) + \frac{\mu}{2}\|y - x\|_2^2$$

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$$= \frac{1}{2}\left(\frac{2}{\sqrt{\mu}}\nabla f(x)^T - \sqrt{\mu}(x^* - x)\right)^T \sqrt{\mu}(x - x^*) =$$

$b + a$ $b - a$ $b^2 - a^2$

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Let $a = \frac{1}{\sqrt{\mu}}\nabla f(x)$ and
$b = \sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}}\nabla f(x)$
Then $a + b = \sqrt{\mu}(x - x^*)$ and
$a - b = \frac{2}{\sqrt{\mu}}\nabla f(x) - \sqrt{\mu}(x - x^*)$

$a^2 = \frac{1}{\mu}\|\nabla f(x)\|^2$

$a^2 - b^2$ $\quad$ $b^2 =$

Any μ-strongly convex differentiable function is a PL-function

100 - 10

$$
f(x) - f(x^*) \leq \frac{1}{2}\left(\frac{1}{\mu}\|\nabla f(x)\|_2^2 - \left\|\sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}}\nabla f(x)\right\|_2^2\right)
$$

100

Any μ-strongly convex differentiable function is a PL-function

$$
f(x) - f(x^*) \leq \frac{1}{2}\left(\frac{1}{\mu}\|\nabla f(x)\|_2^2 - \left\|\sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}}\nabla f(x)\right\|_2^2\right)
$$

$$
f(x) - f(x^*) \leq \frac{1}{2\mu}\|\nabla f(x)\|_2^2,
$$

$$
\|\nabla f(x)\|_2^2 \geq 2\mu\left(f(x) - f^*\right) \leftarrow \text{PL}
$$

Any μ-strongly convex differentiable function is a PL-function

$$
f(x) - f(x^*) \leq \frac{1}{2}\left(\frac{1}{\mu}\|\nabla f(x)\|_2^2 - \left\|\sqrt{\mu}(x - x^*) - \frac{1}{\sqrt{\mu}}\nabla f(x)\right\|_2^2\right)
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Any μ-strongly convex differentiable function is a PL-function

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$$

$$
f(x) - f(x^*) \leq \frac{1}{2\mu}\|\nabla f(x)\|_2^2,
$$

which is exactly the PL condition. It means, that we already have linear convergence proof for any strongly convex function.

Smooth convex case

Smooth convex case

Theorem

сублинейная
! сх-ть !

Consider the Problem

$$f(x) \rightarrow \min_{x \in \mathbb{R}^d}$$

and assume that f is convex and L-smooth, for some $L > 0$.
Let $(x^k)_{k \in \mathbb{N}}$ be the sequence of iterates generated by the gradient descent constant stepsize algorithm, with a stepsize satisfying $0 < \alpha \leq \frac{1}{L}$. Then, for all $x^* \in \operatorname{argmin} f$, for all $k \in \mathbb{N}$ we have that

$$f(x^k) - f^* \leq \frac{\|x^0 - x^*\|^2}{2\alpha k}.$$

Convergence analysis

Лин. парабола $\quad -\alpha\nabla f(x_k)$

- As it was before, we first use smoothness:

$$
\begin{aligned}
\boxed{f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2} \\
= f(x^k) - \alpha\|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2 \\
= f(x^k) - \frac{\alpha}{2}\left(2 - L\alpha\right)\|\nabla f(x^k)\|^2 \\
\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2, \\
\boxed{f(x^k) - f(x^{k+1}) \geq \frac{1}{2L}\|\nabla f(x^k)\|^2} \text{ if } \alpha = \frac{1}{L}
\end{aligned}
\tag{1}
$$

GD →

← $\alpha L \leq 1$

ГРАДИЕНТНЫЙ СПУСК

на каждой итерации сходим

Typically, for the convergent gradient descent algorithm the higher the learning rate the faster the convergence. That is why we often will use $\alpha = \frac{1}{L}$.

Convergence analysis

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f(x^{k+1}) &\leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\
&= f(x^k) - \alpha\|\nabla f(x^k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x^k)\|^2 \\
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  Typically, for the convergent gradient descent algorithm the higher the learning rate the faster the convergence. That is why we often will use $\alpha = \frac{1}{L}$.
- After that we add convexity:

(2)

Convergence analysis

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- After that we add convexity:

диф. крит. вып. I порядка

$$
f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle \tag{2}
$$

Convergence analysis

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&\leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2, \\
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\end{aligned}
\tag{1}
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- After that we add convexity:

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f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle \text{ with } y = x^*, x = x^k \tag{2}
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\boxed{f(x^k) - f^* \leq \langle \nabla f(x^k), x^k - x^* \rangle}
\end{aligned}
\tag{2}
$$

Convergence analysis

- Now we put Equation 2 to Equation 1:

Convergence analysis

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$$f(x^{k+1}) \leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 \leq f^* + \langle \nabla f(x^k), x^k - x^* \rangle - \frac{\alpha}{2}\|\nabla f(x^k)\|^2$$

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$\le a^2 - b^2$

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$$
f(x^{k+1}) \leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^k - x^* - \alpha \nabla f(x^k)\|_2^2\right]
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$$
\begin{aligned}
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&\leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^{k+1} - x^*\|_2^2\right]
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f(x^{k+1}) &\leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^k - x^* - \alpha\nabla f(x^k)\|_2^2\right] \\
&\leq f^* + \frac{1}{2\alpha}\left[\|x^k - x^*\|_2^2 - \|x^{k+1} - x^*\|_2^2\right]
\end{aligned}
$$

$\sum_{k=0}^{N}$

$$
2\alpha\left(f(x^{k+1}) - f^*\right) \leq \|x^k - x^*\|_2^2 - \|x^{k+1} - x^*\|_2^2
$$

Convergence analysis

- Now we put Equation 2 to Equation 1:

$$
\begin{aligned}
f(x^{k+1}) \leq f(x^k) - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 &\leq f^* + \langle \nabla f(x^k), x^k - x^* \rangle - \frac{\alpha}{2}\|\nabla f(x^k)\|^2 \\
&= f^* + \langle \nabla f(x^k), x^k - x^* - \frac{\alpha}{2}\nabla f(x^k) \rangle \\
&= f^* + \frac{1}{2\alpha}\left\langle \alpha \nabla f(x^k), 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)\right\rangle
\end{aligned}
$$

Let $a = x^k - x^*$ and $b = x^k - x^* - \alpha \nabla f(x^k)$. Then $a + b = \alpha \nabla f(x^k)$ and $a - b = 2\left(x^k - x^* - \frac{\alpha}{2}\nabla f(x^k)\right)$.

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$$

- Now suppose, that the last line is defined for some index i and we sum over $i \in [0, k-1]$. Almost all summands will vanish due to the telescopic nature of the sum:

(3)

Convergence analysis

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$$
2\alpha \sum_{i=0}^{k-1} \left(f(x^{i+1}) - f^*\right) \leq \|x^0 - x^*\|_2^2 - \|x^k - x^*\|_2^2 \tag{3}
$$

Convergence analysis

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$$

Convergence analysis

- Due to the monotonic decrease at each iteration $f(x^{i+1}) < f(x^i)$:

$$kf(x^k) \leq \sum_{i=0}^{k-1} f(x^{i+1})$$

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- Now putting it to Equation 3:

$$2\alpha k f(x^k) - 2\alpha k f^* \leq 2\alpha \sum_{i=0}^{k-1} \left(f(x^{i+1}) - f^*\right) \leq \|x^0 - x^*\|_2^2$$

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$$

$$
f(x^k) - f^* \leq \frac{\|x^0 - x^*\|_2^2}{2\alpha k}
$$

$\sim \frac{1}{k}$

Convergence analysis

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$$2\alpha kf(x^k) - 2\alpha kf^* \leq 2\alpha \sum_{i=0}^{k-1} \left(f(x^{i+1}) - f^*\right) \leq \|x^0 - x^*\|_2^2$$

$$f(x^k) - f^* \leq \frac{\|x^0 - x^*\|_2^2}{2\alpha k} \leq \frac{L\|x^0 - x^*\|_2^2}{2k}$$

Summary

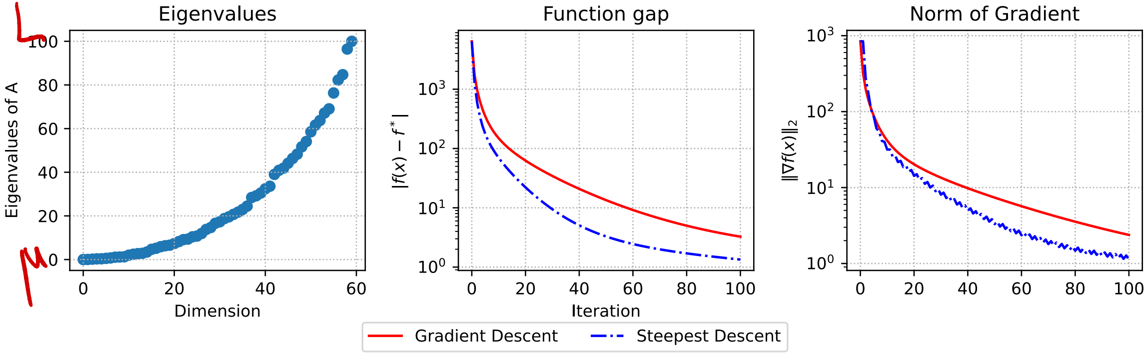
Gradient Descent: $$\min_{x \in \mathbb{R}^n} f(x) \qquad x^{k+1} = x^k - \alpha^k \nabla f(x^k)$$

smooth (non-convex)	smooth & convex	smooth & strongly convex (or PL)
$\|\nabla f(x^k)\|^2 \sim \mathcal{O}\left(\frac{1}{k}\right)$	$f(x^k) - f^* \sim \mathcal{O}\left(\frac{1}{k}\right)$	$\|x^k - x^*\|^2 \sim \mathcal{O}\left(\left(1 - \frac{\mu}{L}\right)^k\right)$
$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon}\right)$	$k_\varepsilon \sim \mathcal{O}\left(\frac{1}{\varepsilon}\right)$	$k_\varepsilon \sim \mathcal{O}\left(\varkappa \log \frac{1}{\varepsilon}\right)$

Numerical experiments

$$f(x) = \frac{1}{2}x^T A x - b^T x \to \min_{x \in \mathbb{R}^n}$$

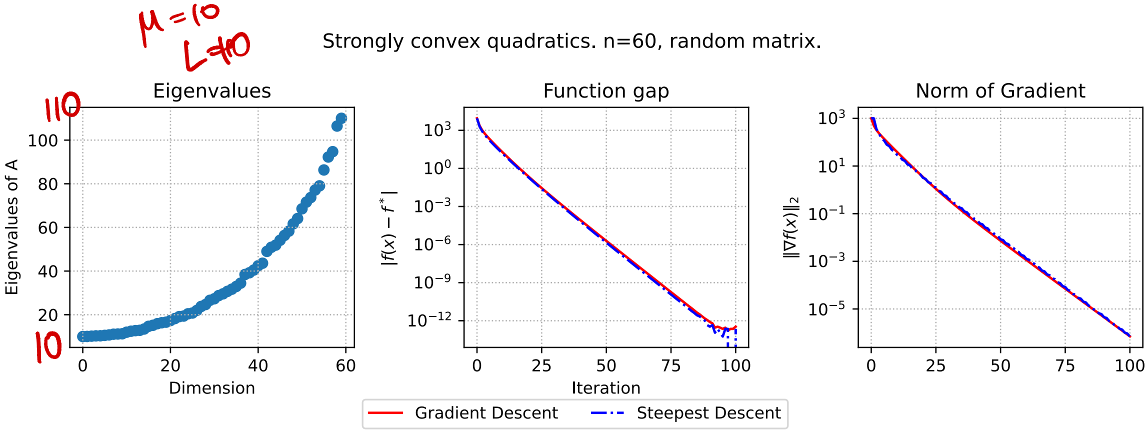
Convex quadratics. n=60, random matrix.



Numerical experiments

$$
f(x) = \frac{1}{2}x^T A x - b^T x \to \min_{x \in \mathbb{R}^n}
$$

Strongly convex quadratics. n=60, random matrix.



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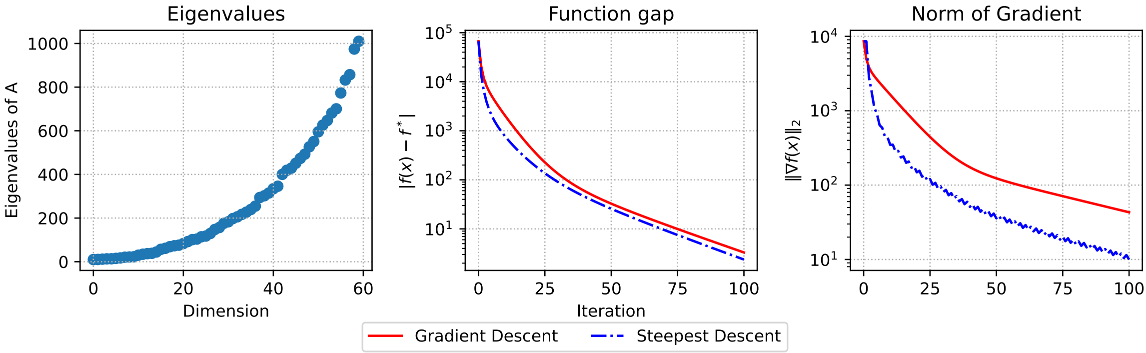
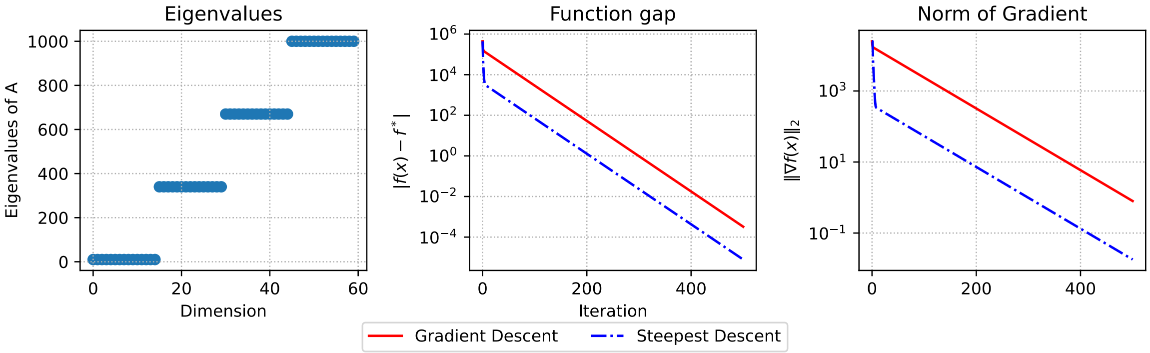


Numerical experiments

$$
f(x) = \frac{1}{2}x^T A x - b^T x \to \min_{x \in \mathbb{R}^n}
$$

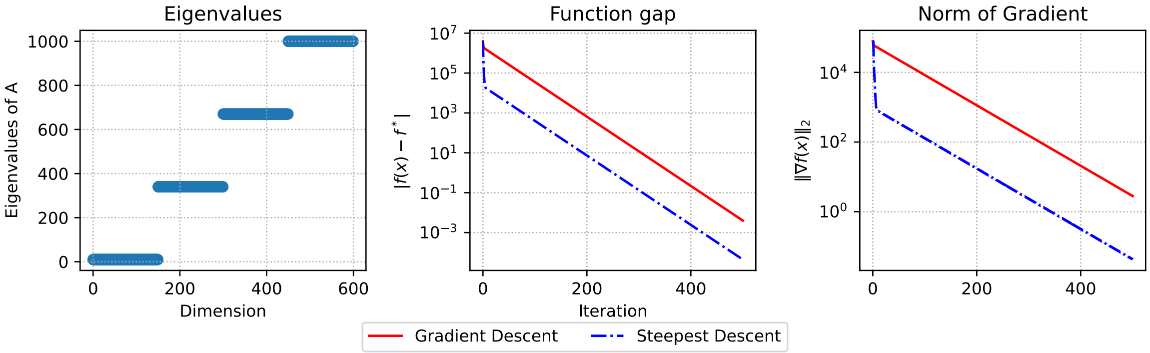
Strongly convex quadratics. n=60, clustered matrix.



Numerical experiments

$$f(x) = \frac{1}{2}x^T A x - b^T x \to \min_{x \in \mathbb{R}^n}$$

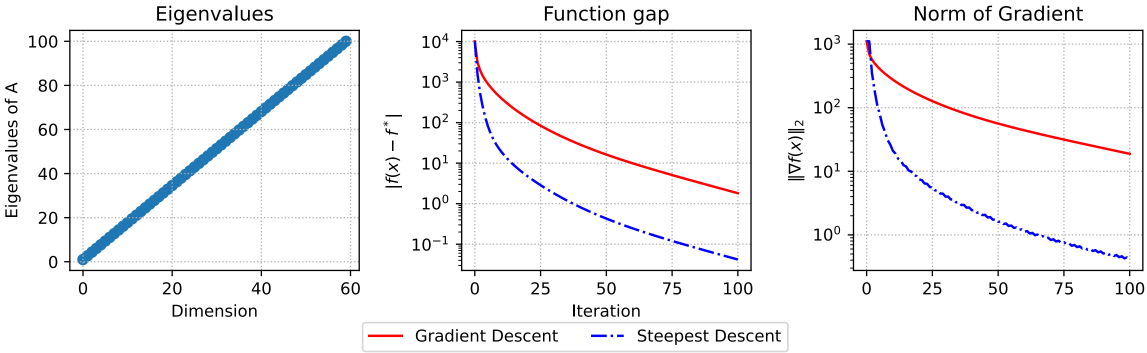
Strongly convex quadratics. n=600, clustered matrix.



Numerical experiments

$$f(x) = \frac{1}{2}x^T A x - b^T x \to \min_{x \in \mathbb{R}^n}$$

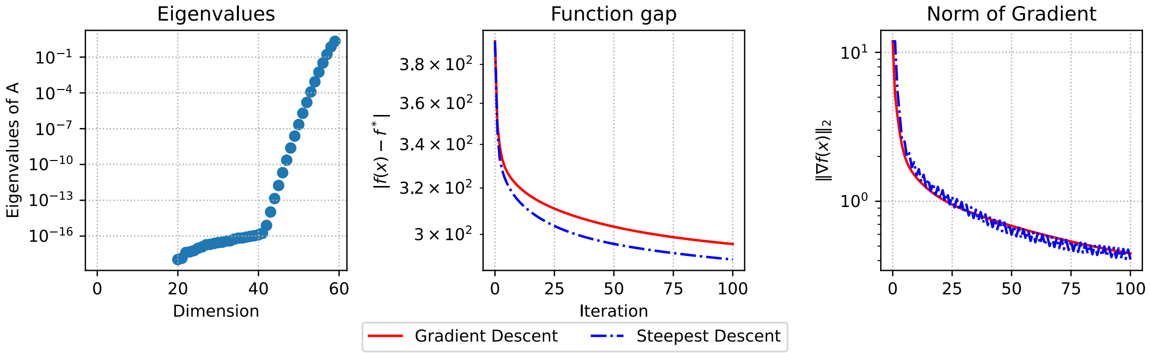
Strongly convex quadratics. n=60, uniform spectrum matrix.



Numerical experiments

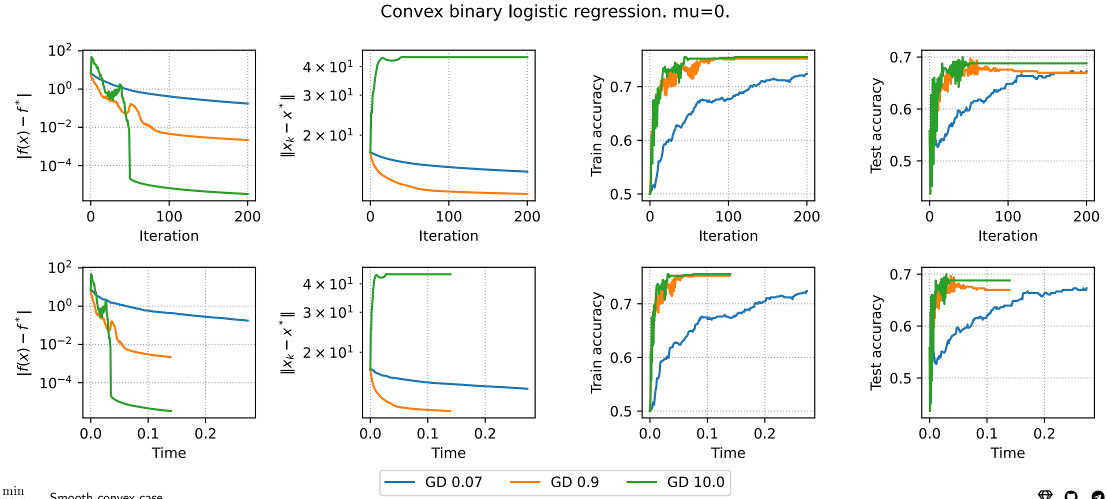
$$f(x) = \frac{1}{2}x^T A x - b^T x \to \min_{x \in \mathbb{R}^n}$$

Strongly convex quadratics. n=60, Hilbert matrix.



Numerical experiments

$$f(x) = \frac{\mu}{2}\|x\|_2^2 + \frac{1}{m}\sum_{i=1}^{m}\log(1+\exp(-y_i\langle a_i, x\rangle)) \to \min_{x\in\mathbb{R}^n}$$



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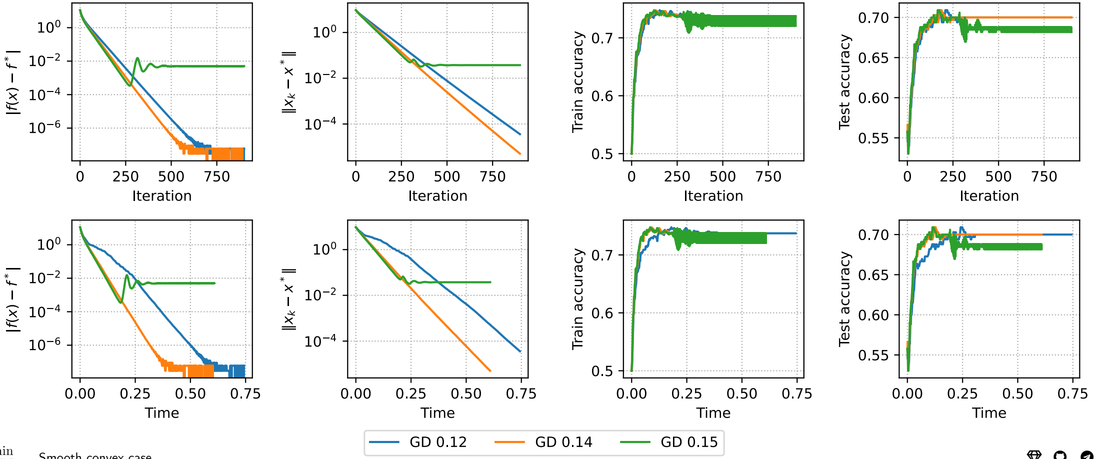
Strongly convex binary logistic regression. mu=0.1.



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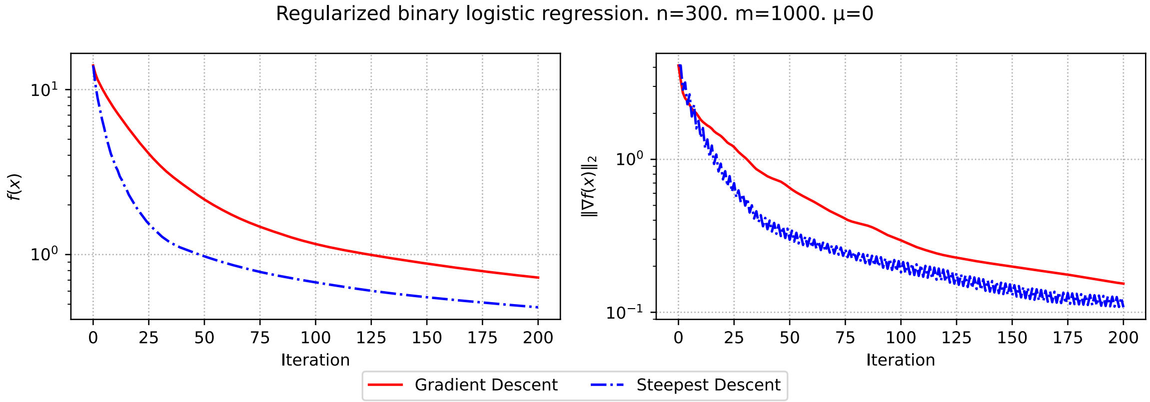
$$f(x) = \frac{\mu}{2}\|x\|_2^2 + \frac{1}{m}\sum_{i=1}^{m}\log(1+\exp(-y_i\langle a_i, x\rangle)) \to \min_{x\in\mathbb{R}^n}$$



Numerical experiments

$$
f(x) = \frac{\mu}{2}\|x\|_2^2 + \frac{1}{m}\sum_{i=1}^{m}\log(1+\exp(-y_i\langle a_i, x\rangle)) \to \min_{x\in\mathbb{R}^n}
$$